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Note

## Independent finite sums in graphs defined on the natural numbers<sup>1</sup>

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### Abstract

In this note we present several results related to conjectures of Erdős and Hajnal on the existence of independent sets with good arithmetic properties in a locally sparse graph whose vertices are natural numbers. In particular, we prove that if  $k, \ell \geq 2$  and a graph  $G$  defined on the natural numbers contains no copies of the complete graph on  $k$  vertices, then there exists a subset  $A \subseteq \mathbb{N}$  such that the set  $\text{FS}_{\leq \ell}(A) = \{\sum_{i \in I} a_i : I \subseteq \mathbb{N} \text{ and } |I| \leq \ell\}$ , is independent in  $G$ , which settles Erdős' question in the affirmative. © 1998 Elsevier Science B.V.

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In 1995 Paul Erdős conjectured that for every  $k$  there exists  $n_0$  such that for every  $n \geq n_0$  the following holds: for each graph  $G$  with vertex set  $\{1, 2, \dots, n\}$  which contains no copies of the complete graph  $K_k$  on  $k$  vertices there exists  $A \subseteq \{1, \dots, n\}$  such that all finite sums of different elements of  $A$  span in  $G$  an independent set. An infinite version of this problem was stated by András Hajnal, who asked if, for a graph  $G$  defined on the set of natural numbers, there exists an infinite set  $A$  with the above property. Hajnal's question has been recently answered in the negative by Deuber, Gunderson, Hindman and Strauss in [1]. In the same paper the authors prove also that disjoint and bipartite versions of Hajnal's conjecture hold (see [1] for details). The main result of this note, Theorem 5, asserts that a finite (or, more precisely, 'semi-infinite') version of Hajnal's conjecture remains true as well, which, in particular, settles Erdős' conjecture in the affirmative. We also provide a simple proof for a bipartite version of Hajnal's question, stated as Theorem 7.

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We shall use the following definitions and notation. Let  $A = \{a_1, a_2, \dots\} \subseteq \mathbb{N}$ , where throughout the note all elements of subsets are given in the increasing order, i.e.,  $a_1 < a_2 < \dots$ . For such an  $A$  by  $[A]^k$  we denote the set of all its subsets with  $k$  elements, put

$$\text{FS}_{\leq k}(A) = \left\{ \sum_{i \in I} a_i : I \subseteq \mathbb{N} \text{ and } |I| \leq k \right\},$$

and  $\text{FS}(A) = \bigcup_{k \geq 1} \text{FS}_{\leq k}(A)$ .

Let  $I, J$  be two nonempty finite subsets of natural numbers, i.e.,  $I, J \subseteq_{\text{fin}} \mathbb{N}$ . We write  $I \prec J$  whenever  $\max I < \min J$ . More generally, we characterize the mutual position of  $I$  and  $J$  by introducing a sequence  $\text{mix}(I, J)$  of 0's, 1's and 2's called the *mixing type* of a pair  $(I, J)$ . To find the mixing type of a pair  $(I, J)$  we proceed as follows. Let  $I \cup J = \{r_1, r_2, \dots, r_s\}$  and for  $t = 1, 2, \dots, s$ , let

$$w_t = \begin{cases} 0, & \text{if } r_t \in I \setminus J, \\ 1, & \text{if } r_t \in J \setminus I, \\ 2, & \text{if } r_t \in I \cap J. \end{cases}$$

Now, to obtain  $\text{mix}(I, J)$ , replace each block of consecutive same elements in  $w_1 w_2 \dots w_s$  by just one representative. (Thus, for example, for sets  $I_0 = \{1, 3, 4, 5, 7\}$  and  $J_0 = \{2, 5, 6, 7\}$  we have  $\text{mix}(I_0, J_0) = 010212$ .) Note that if  $\ell = \ell(I, J)$  is the *length* of the mixing type  $\text{mix}(I, J)$  then there exists a natural partition of  $I \cup J$  into  $\ell$  sets  $L_1, L_2, \dots, L_\ell$ , such that  $L_1 \prec L_2 \prec \dots \prec L_\ell$  and for every  $s = 1, 2, \dots, \ell$ , the set  $L_s$  is contained in precisely one of the three sets  $I \setminus J$ ,  $J \setminus I$  and  $I \cap J$ . We call the above partition the *proper decomposition* of  $(I, J)$ .

Furthermore,  $M_{\leq s}$  will stand for the set of all mixing types of length not larger than  $s$  and

$$[\text{FS}(A)]_{\leq s}^2 = \left\{ \left\{ \sum_{i \in I} a_i, \sum_{i \in J} a_i \right\} : a_i \in A \text{ and } \ell(I, J) \leq s \right\}.$$

Finally, for an infinite set  $B = \{b_1, b_2, \dots\}$  we write  $B \subseteq \text{FS}(A)$  if there are subsets  $I_1 \prec I_2 \prec \dots$  such that for every  $k = 1, 2, \dots$ , we have  $b_k = \sum_{i \in I_k} a_i$ .

Our argument is based on the following well-known result of Milliken [4] and Taylor [5]. Let us remark that the Milliken–Taylor theorem has been applied for somewhat similar ‘mixing types’ problems in the paper of Deuber and Rothschild [2].

**Theorem 1.** *Let  $k \in \mathbb{N}$  and let  $A$  be an infinite set of natural numbers. Then for every coloring of  $[\text{FS}(A)]^k$  with a finite number of colors there exists  $B \subseteq \text{FS}(A)$ ,  $B = \{b_1, b_2, \dots\}$ , such that the set  $\{\sum_{i \in I_1} b_i, \sum_{i \in I_2} b_i, \dots, \sum_{i \in I_k} b_i\}$  has the same color for every choice of  $I_1 \prec I_2 \prec \dots \prec I_k$ .*

From the above result we deduce the following lemma.

**Lemma 2.** Let  $M = \{m_1, m_2, \dots, m_d\}$  be any finite set of mixing types. Then for every infinite set of natural numbers  $A$  and every finite coloring  $\chi$  of  $[\text{FS}(A)]^2$  there exists  $B \subseteq \text{FS}(A)$ ,  $B = \{b_1, b_2, \dots\}$ , such that the set

$$\left\{ \left\{ \sum_{i \in I} b_i, \sum_{i \in J} b_i \right\} : b_i \in B \text{ and } \text{mix}(I, J) = m_s \right\}$$

is monochromatic for every  $s = 1, 2, \dots, d$ .

**Proof.** We shall use the induction with respect to  $d = |M|$ . Let  $d = 1$ , i.e.,  $M = \{m_1\}$  for some mixing type  $m_1$  such that  $\ell(m_1) = \ell_1$ . We define an auxiliary coloring  $\chi_1$  of  $[\text{FS}(A)]^{\ell_1}$  setting

$$\chi_1 \left( \left\{ \sum_{i \in L_1} a_i, \dots, \sum_{i \in L_{\ell_1}} a_i \right\} \right) = \chi \left( \left\{ \sum_{i \in I} a_i, \sum_{i \in J} a_i \right\} \right),$$

whenever  $\text{mix}(I, J) = m_1$  and  $L_1 \cup \dots \cup L_{\ell_1}$  is the proper decomposition  $I \cup J$ , and extend  $\chi_1$  to  $[\text{FS}(A)]^{\ell_1}$  in an arbitrary way. From the Milliken–Taylor theorem there exists a set  $B \subseteq \text{FS}(A)$ ,  $B = \{b_1, b_2, \dots\}$ , such that all pairs  $\{\sum_{i \in I} b_i, \sum_{i \in J} b_i\}$  with  $\text{mix}(I, J) = m_1$  are colored with the same color in  $\chi$ .

Now suppose that the assertion holds for every set  $M'$  consisting of  $d$  elements, and let  $M = \{m_1, \dots, m_{d+1}\}$ . By the induction hypothesis one can find a set  $B \subseteq \text{FS}(A)$ ,  $B = \{b_1, b_2, \dots\}$ , such that for  $k = 1, 2, \dots, d$ , all pairs  $\{\sum_{i \in I} b_i, \sum_{i \in J} b_i\}$  with  $\text{mix}(I, J) = m_k$  are colored with the same color in  $\chi$ . Now our previous argument applied to sets  $\{m_{d+1}\}$  and  $B$  implies the existence of  $C = \{c_1, c_2, \dots\}$ , such that  $C \subseteq \text{FS}(B)$ , and thus  $C \subseteq \text{FS}(A)$ , and furthermore all pairs  $\{\sum_{i \in I} c_i, \sum_{i \in J} c_i\}$  with  $\text{mix}(I, J) = m_k$  are monochromatic for  $k = 1, 2, \dots, d + 1$ .  $\square$

Our next result, as well as the following Theorem 3\*, can be derived from an infinite version of the Hales–Jewett theorem proved by Furstenberg and Katznelson [3], nonetheless we have decided to present its “elementary” proof which invokes neither ultrafilters nor dynamical system tools.

**Theorem 3.** Let  $A$  be an infinite subset of  $\mathbb{N}$  and let  $(s_n)_{n=1}^\infty$  be an arbitrary sequence of natural numbers. Then for any coloring  $\chi$  of  $[\text{FS}(A)]^2$  with finite number of colors there exists  $B \subseteq \text{FS}(A)$ ,  $B = \{b_1, b_2, \dots\}$ , such that for every finite subsets  $I, I', J, J'$  which consist of natural numbers not smaller than  $n$  and are such that for  $\text{mix}(I, J) = \text{mix}(I', J')$  and  $\ell(I, J) \leq s_n$  we have

$$\chi \left( \left\{ \sum_{i \in I} b_i, \sum_{i \in J} b_i \right\} \right) = \chi \left( \left\{ \sum_{i \in I'} b_i, \sum_{i \in J'} b_i \right\} \right).$$

**Proof.** We recursively construct an infinite sequence of sets  $(B^k)_{k=1}^\infty$ ,  $B^k = \{b_1^k, b_2^k, \dots\}$ , such that  $B^1 \subseteq \text{FS}(A)$ ,  $B^{k+1} \subseteq \text{FS}(B^k \setminus \{b_1^k\})$  for  $k = 1, 2, \dots$ , and for every finite subsets

$I, I', J, J'$  which are such that  $\text{mix}(I, J) = \text{mix}(I', J')$  and  $\ell(I, J) \leq s_k$  we have

$$\chi \left( \left\{ \sum_{i \in I} b_i^k, \sum_{i \in J} b_i^k \right\} \right) = \chi \left( \left\{ \sum_{i \in I'} b_i^k, \sum_{i \in J'} b_i^k \right\} \right).$$

In order to find the set  $B^1$  it is enough to apply Lemma 2 to sets  $M_{\leq s_1}$  and  $A$ . Furthermore, once the sets  $B^1, \dots, B^k$  have been already constructed, we can use Lemma 2 for sets  $M_{\leq s_{k+1}}$  and  $B^k \setminus \{b_1^k\}$  and obtain a set  $B^{k+1}$  with the desired properties.

Now, to complete the proof of Theorem 3 it is enough to take  $B = \{b_1^1, b_1^2, \dots\}$ .  $\square$

Using the same idea one can in fact show a slightly stronger result stated below as Theorem 3\*. Its proof however is longer and more involved, mainly because of purely technical details, so we have decided to omit it here.

**Theorem 3\*** *Let  $A$  be an infinite subset of  $\mathbb{N}$  and let  $(s_n)_{n=1}^\infty$  be an arbitrary sequence of natural numbers. Then for any coloring  $\chi$  of  $[\text{FS}(A)]^2$  with finite number of colors there exists  $B \subseteq \text{FS}(A)$ ,  $B = \{b_1, b_2, \dots\}$ , such that*

$$\chi \left( \left\{ \sum_{i \in I} b_i, \sum_{i \in J} b_i \right\} \right) = \chi \left( \left\{ \sum_{i \in I'} b_i, \sum_{i \in J'} b_i \right\} \right)$$

*provided  $I, I', J, J' \subseteq_{\text{fin}} \mathbb{N}$ , are such that  $I \cap \{1, 2, \dots, n-1\} = I' \cap \{1, 2, \dots, n-1\}$ ,  $J \cap \{1, 2, \dots, n-1\} = J' \cap \{1, 2, \dots, n-1\}$ ,  $\text{mix}(I, J) = \text{mix}(I', J')$  and  $\ell(I, J) \leq s_n$ .*

In order to apply Theorems 3 and 3\* we need the following simple observation.

**Fact 4.** *For every mixing type  $m$  and every  $k \geq 2$  there exist sets  $I_1, \dots, I_k \subseteq_{\text{fin}} \mathbb{N}$  such that  $m = \text{mix}(I_i, I_j)$  for every  $1 \leq i < j \leq k$ .*

**Proof.** We verify the assertion using elementary induction on the length of  $m$ . If  $m$  is of length one, i.e.,  $m = 2$ , one can just take  $I_1 = \dots = I_k$ . Let us suppose that  $m = w_1 w_2 \dots w_{\ell+1}$  and let  $I'_1, I'_2, \dots, I'_k \subseteq_{\text{fin}} \mathbb{N}$  be such that for every  $1 \leq i < j \leq k$  we have  $\text{mix}(I'_i, I'_j) = w_1 \dots w_\ell$ . Furthermore, let  $J_1, \dots, J_k \subseteq_{\text{fin}} \mathbb{N}$  be such that

$$\bigcup_{i=1}^k I'_i \prec J_1 \prec J_2 \prec \dots \prec J_k.$$

Now, for  $i = 1, 2, \dots, k$ , set

$$I_i = \begin{cases} I'_i \cup J_1, & \text{if } w_{\ell+1} = 2, \\ I'_i \cup J_i, & \text{if } w_{\ell+1} = 1, w_\ell = 0, \\ I'_i \cup J_{k+1-i}, & \text{if } w_{\ell+1} = 0, w_\ell = 1, \\ I'_i \cup \bigcup_{j=1}^i J_j, & \text{if } w_{\ell+1} = 1, w_\ell = 2, \\ I'_i \cup \bigcup_{j=1}^{k+1-i} J_j, & \text{if } w_{\ell+1} = 0, w_\ell = 2. \end{cases}$$

Then,  $\text{mix}(I_i, I_j) = w_1 \dots w_{\ell+1}$  and the assertion follows.  $\square$

The above fact, together with Theorem 3, leads to the main result of this note.

**Theorem 5.** *Let  $G$  be a graph with the vertex set  $\mathbb{N}$  which does not contain a copy of the complete graph on  $k$  vertices for some  $k \geq 3$ . Then for every sequence of natural numbers  $(s_n)_{n=1}^\infty$  there exists an infinite set  $A = \{a_1, a_2, \dots\}$  such that for every  $n \geq 1$  no pair from  $[\text{FS}(\{a_n, a_{n+1}, \dots\})]_{\leq s_n}^2$  is an edge of  $G$ .*

**Proof.** Let  $A = \{a_1, a_2, \dots\}$  be the set whose existence follows from Theorem 3 applied to the coloring  $[\mathbb{N}]^2 = G \cup G^c$ , and let us suppose that for some  $n$  a pair from  $[\text{FS}(\{a_n, a_{n+1}, \dots\})]_{\leq s_n}$  is an edge of  $G$ . Then, there exists a mixing type  $m$  of length not larger than  $s_n$ , such that for every pair of sets  $I, J \subseteq \{n, n+1, \dots\}$  with  $\text{mix}(I, J) = m$  the pair  $\{\sum_{i \in I} a_i, \sum_{i \in J} a_i\}$  is an edge of  $G$ .

Now apply Fact 4 and choose sets  $I_1, \dots, I_k \subseteq \{n, n+1, \dots\}$  in such a way that  $\text{mix}(I_i, I_j) = m$  for every  $1 \leq i < j \leq k$ . Then vertices  $\sum_{i \in I_1} a_i, \dots, \sum_{i \in I_k} a_i$  span in  $G$  a complete graph which contradicts the assumption on  $G$ .  $\square$

**Corollary 6.** *Let  $k, \ell$  be natural numbers and  $G$  be a graph with vertex set  $\mathbb{N}$  which contains no complete subgraphs of order  $k$ . Then there exists an infinite set  $A = \{a_1, a_2, \dots\}$  such that for every  $I, J \subseteq \mathbb{N}$  with  $\ell(I, J) \leq \ell$  vertices  $\sum_{i \in I} a_i$  and  $\sum_{i \in J} a_i$  are not joined by an edge. In particular, the set  $\text{FS}_{\leq \ell}(A)$  is independent.*

**Proof.** It is enough to apply Theorem 5 with  $s_1 = \ell$ .  $\square$

Let us remark that the above statement is not valid under the weaker assumption that  $G$  contains no complete graphs of infinite size. Indeed, the graph  $G_0$  with edge set  $\{\{v, w\} : v < w \leq 2v\}$  contains no infinite complete subgraphs, but clearly for every  $v, w \in \mathbb{N}$ ,  $v < w$ ,  $w$  is adjacent to  $v + w$  in  $G_0$ .

We conclude the note by showing that if one prohibits in  $G$  large complete *bipartite* subgraphs then the answer to Hajnal's question is positive. A different proof of this result, based on an ultrafilters approach, can be found in Deuber, Gunderson, Hindman and Strauss [1].

**Theorem 7.** *Let  $G$  be a graph with the vertex set  $\mathbb{N}$  which contains no copies of the complete balanced bipartite graph on  $2k$  vertices for some natural  $k$ . Then there exists an infinite set  $A \subseteq \mathbb{N}$  such that the set  $\text{FS}(A)$  is independent.*

**Proof.** Set  $s_n = n + 1$  and, similarly as in the proof of Theorem 5, apply Theorem 3\* for  $\mathbb{N} = G \cup G^c$  and the sequence  $(s_n)_{n=1}^\infty$  to find a 'monochromatic' sequence  $A = \{a_1, a_2, \dots\}$ . Let us suppose that for some  $I, J \subseteq \text{fin } \mathbb{N}$  the pair  $\{\sum_{i \in I} a_i, \sum_{i \in J} a_i\}$  is an edge of  $G$ . Since  $s_n = n + 1$  we can find  $n_0 \in \mathbb{N}$  such that  $\text{mix}(I, J) = s_{n_0} = n_0 + 1$  and so

$$m = \text{mix}(I \setminus \{1, 2, \dots, n_0 - 1\}, J \setminus \{1, 2, \dots, n_0 - 1\}) \geq 2.$$

Thus, using Fact 4 one can find  $2k$  different sets  $I'_1, \dots, I'_k, J'_1, \dots, J'_k \subseteq_{\text{fin}} \mathbb{N} \setminus \{1, 2, \dots, n_0 - 1\}$ , such that  $\text{mix}(I'_i, J'_j) = m$  for every  $1 \leq i, j \leq k$ . Now, it is enough to set for  $i = 1, 2, \dots, k$ ,

$$I_i = (I \cap \{1, 2, \dots, n_0 - 1\}) \cup I'_i,$$

$$J_i = (J \cap \{1, 2, \dots, n_0 - 1\}) \cup J'_i,$$

and observe that elements  $\{\sum_{i \in I_1} a_i, \dots, \sum_{i \in I_k} a_i, \sum_{i \in J_1} a_i, \dots, \sum_{i \in J_k} a_i\}$  span in  $G$  a copy of the complete balanced bipartite graph on  $2k$  vertices contradicting the assumption on  $G$ .  $\square$

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